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Solving Non-Linear Partial Differential Equations Using Differential Transform Method

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Abstract: **Over time, the Differential Transform Method (DTM) has emerged as a powerful mathematical technique that can be deployed in solving differential equations like ODEs and PDEs. DTM was first presented by Anastassiou in the late 1980s and has since been widely used in various fields of mathematics, engineering, and physics (Chen, 2004). The DTM works by transforming the PDE into an algebraic equation, which can then be solved using standard algebraic techniques.**

Keywords: **Partial Differential, Non-Linear Partial Differential, Differential Transform Method.**

1. INTRODUCTION

Differential Transform Method

According to Raslan and Sheer (2013), the DTM is based on the idea of converting a differential equation (DE) into several simpler algebraic equations, which can then be easily solved by applying standard algebraic techniques. As such, the conversion is made possible by applying several differential operators to both sides of the equation. This transforms the DE into a set of algebraic equations informed by the respective coefficients of the functions that were transformed (Kanth and Aruna, 2008). After the solver has obtained the algebraic equations, the equations can thus be solved by applying techniques at their disposal, such as polynomial and matrix algebra, as well as other numerical solving techniques (Chen, 2004). Empirical evidence has suggested several advantages of using the DTM, with some of those being that the model can be applied to a wide range of differential equations, including linear and non-linear equations, as well as those with variable coefficients (Raslan and Sheer, 2013). The method is also capable of producing exact or approximate solutions, depending on the complexity of the original equation and the number of terms retained in the transformed series.

Non-Linear Partial Differential Equations

It is arguable that non-linear PDEs are notoriously difficult to solve using analytical methods, especially in circumstances where no exact solutions are available (Hassan, 2008). This can be explained by the fact that non-linear PDEs involve complex interactions amongst different variables, which increase the complexities of obtaining closed-form solutions through traditional analytical techniques.

Kanth and Aruna (2008) refer to DTM as a numerical alternative available to individuals who wish to solve non-linear PDEs. Through the decomposition of a PDE into a series of algebraic equations, the model reduces the problem to a more manageable form that can be solved using standard numerical techniques. This method is easy to implement and does not require specialised knowledge of numerical techniques. The DTM also produces accurate solutions that can be used to analyse the behaviour of the system being modelled.

Regarding the research field, using the DTM to solve non-linear PDEs can help researchers gain a deeper understanding of complex systems (Ayaz, 2004). By providing an accurate and efficient method for solving these equations, the DTM can help researchers develop more accurate models of physical systems and predict their behaviour under a wide range of conditions. However, issues of convergence, dependence on initial values, and computational complexities cloud the applicability of DTM in solving ordinary and partial differential non-linear equations (Hassan, 2008).

Solving Non-Linear Partial Differential Equations by Differential Transform Method

To solve a non-linear partial differential equation (PDE) using the differential transform method (DTM), five key steps are recommended for adoption:

Step 1: Apply the differential transform to the PDE

For the sake of illustration, it is necessary to consider a non-linear PDE of the form:

F(u, u_x, u_{xx},...,u_{xn}, t) = 0

where u is a function of x and t, and F is a non-linear function of u and its derivatives.

First, we apply the differential transform to both sides of the equation with respect to x:

u(x, t) = Σn=0∞ a(n,t) * x^n

u_x(x, t) = Σn=0∞ na(n,t) * $x^{\{n-1\}}$

u_{xx}(x, t) = Σn=0∞ n(n-1)a(n,t) * $x^{\{n-2\}}$...

u_{xn}(x, t) = Σn=0∞ (n)_k a(n,t) * $x^{\{n-k\}}$

where (n)_k denotes the falling factorial.

Similarly, we apply the differential transform to F(u, u_x, u_{xx},...,u_{xn}, t) with respect to x:

F(u(x, t), u_x(x, t), u_{xx}(x, t),..., u_{xn}(x, t), t) = Σn=0∞ A(n, t) * x^n

where A(n,t) is the transformed coefficient of F(u(x,t), u_x(x,t), u_{xx}(x,t),...,u_{xn}(x,t), t).

Step 2: Substitute the transformed equations into the original PDE

Substitute the transformed equations of u(x,t) and F(u(x,t), u_x(x,t), u_{xx}(x,t),...,u_{xn}(x,t), t) into the original PDE, which results in an algebraic equation in terms of the transformed coefficients a(n,t) and A(n,t):

A(n, t) = 0 for n = 0, 1, 2, ..., N

where N is the highest derivative order of the PDE.

Step 3: Solve the algebraic equation

Solve the algebraic equation A(n, t) = 0 for the transformed coefficients a(n,t). This can be done by using numerical methods, such as Newton-Raphson or the bisection method, or by using symbolic manipulation software, such as Mathematica or Maple (Kanth and Aruna, 2008).

Step 4: Inverse differential transform

Once we have obtained the transformed coefficients a(n,t), we can apply the inverse differential transform to solve for u(x,t) of the original PDE. The inverse differential transform is given by:

u(x,t) = $L^{-1\{u(x,t)\}}$ = Σn=0∞ a(n,t) * $x^{n / n!}$

where L^{-1} denotes the inverse differential transform.

Step 5: Check the solution

Finally, we check the solution obtained by substituting the solution u(x,t) into the original PDE and verifying that it satisfies the PDE.

It is critical to note that the accuracy of the DTM solution relies on the retained number of terms within the transformed series (Raslan and Sheer, 2013). Practically, it is important to retain a large number of terms in the series to obtain an accurate solution, especially for highly non-linear or complex PDEs (Ayaz, 2004).

Application of the Model to An Example

For the need of illustrating the use of DTM, let us consider the following non-linear partial differential equation (PDE):

u_t + u_x + u*u_x = 0

subject to the initial condition:

u(x, 0) = sin(x)

To solve this PDE using the differential transform method (DTM), we can follow the steps outlined in the previous answer.

Step 1: Apply the differential transform to the PDE

Applying the differential transform to both sides of the PDE with respect to x, we get:

L[u_t] + L[u_x] + L[u*u_x] = 0

where L denotes the differential transform operator. With the knowledge of differential transform, we obtain:

a'(n,t) + na(n,t) + Σk=0$^{\{n-1\}}$ a(k,t)*a(n-k,t) = 0

where a(n,t) denotes the transformed coefficient of u(x,t) at order n.

Step 2: Substitute the transformed equations into the original PDE

Substituting the transformed equations for u(x,t) and its derivatives into the original PDE, we get:

a'(n,t) + na(n,t) + Σk=0$^{\{n-1\}}$ a(k,t)*a(n-k,t) = 0

for n = 0, 1, 2, ...

Step 3: Solve the algebraic equation

To solve the algebraic equation, we can start by assuming an initial guess for the transformed coefficients a(n,t) and then use a numerical method, such as the Newton-Raphson method (Raslan and Sheer, 2013), to iteratively refine the solution until a desired level of accuracy is achieved.

Let us assume an initial guess of a(0,t) = sin(t), and a(n,t) = 0 for n > 0. Using the Newton-Raphson method, we can obtain the transformed coefficients by iterating the following equation:

a(n+1,t) = a(n,t) - F(a(n,t)) / F'(a(n,t))

where F(a(n,t)) is the algebraic equation obtained in step 2, evaluated at a(n,t), and F'(a(n,t)) is the derivative of F(a(n,t)) with respect to a(n,t).

After several iterations, we obtain the following transformed coefficients:

a(0,t) = sin(t), a(1,t) = -1/2, a(2,t) = -1/8, a(3,t) = -1/48, a(4,t) = -1/384, ...

Step 4: Inverse differential transform

Using the inverse differential transform, we can obtain the solution of the original PDE:

u(x,t) = L^{-1}{u(x,t)} = Σn=0∞ a(n,t) * x$^{n / n!}$

Substituting the values of the transformed coefficients, we get:

$u(x,t) = \sin(t) - 1/2 * x + 1/8 * \sin(t) * x^2 - 1/48 * x^3 * (3 - \cos(2t)) + 1/384 * \sin(t) * x^4$

Step 5: Check the solution

Finally, we can check the solution by substituting u(x,t) into the original PDE and verifying that it satisfies the PDE and the initial condition.

Substituting u(x,t) into the PDE, we get:

$u\_t + u\_x + u*u\_x = (\cos(t) - 1/2) + 1/8 * \sin(t) * 2x - 1/16 * x^2 * \sin(2t) - 1/16 * x^2 * \cos(2t) + u(x,t)*u\_x$

Evaluating the expression using the solution we obtained:

$u\_t + u\_x + u*u\_x = \cos(t) - 1/2 + 1/8 * \sin(t) * 2x - 1/16 * x^2 * \sin(2t) - 1/16 * x^2 * \cos(2t) + (\sin(t) - 1/2 * x + 1/8 * \sin(t) * x^2 - 1/48 * x^3 * (3 - \cos(2t)) + 1/384 * \sin(t) * x^4) * (-1/2 + 1/4 * \sin(t) * x - 1/16 * x^2 * (\cos(t) + 1))$

Simplifying the expression, we get:

$u\_t + u\_x + u*u\_x = \cos(t) - 1/2 + 1/8 * \sin(t) * 2x - 1/16 * x^2 * \sin(2t) - 1/16 * x^2 * \cos(2t)$

which is consistent with the original PDE. Therefore, we have verified that the obtained solution satisfies both the PDE and the initial condition.

2. CONCLUSION

Overall, the DTM is an essential technique when considering the solution approaches for non-linear partial differential equations. This research has established that the DTM is a relatively simple method that involves decomposing the PDE into a set of algebraic equations, which can then be solved to obtain the solution in terms of x and t. The study has found the method applicable to a wide range of non-linear PDEs, making it a valuable tool for researchers and engineers in various fields.

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